Green's Function (6A)

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Green's Function

Zero State IVP

$$y'' + P(x)y' + Q(x)y = 0$$
$$y'(x_0) = y_1$$
$$y(x_0) = y_0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y'(x_0) = 0$$
$$y(x_0) = 0$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$u_1'(x) = \frac{W_1}{W} = -\frac{y_2(x) f(x)}{W}$$

$$\begin{vmatrix} f & y_2 \\ 0 & y_2' \end{vmatrix} = W$$

$$u_2'(x) = \frac{W_2}{W} = \frac{y_1(x) f(x)}{W}$$

$$\begin{vmatrix} y_1 & f \\ y_1' & 0 \end{vmatrix} = W$$

$$y_p(x_0) = 0$$
$$y_p'(x_0) = 0$$

Anti-derivatives of $u_1(x)$ & $u_2(x)$

$$u_1'(x) = \frac{W_1}{W} = -\frac{y_2(x) f(x)}{W}$$

$$u_1(x) = \int u_1'(x)\, dx$$

$$\text{anti-derivative} = \int -\frac{y_2(t) f(t)}{W(t)}\, dt + c_1$$

$$= \int_{x_0}^{x} -\frac{y_2(t) f(t)}{W(t)}\, dt$$

$$u_1(x_0) = 0$$

➡ $u_1(x_0)\, y_1(x_0) = 0$

➡ $u_1(x_0)\, y_1'(x_0) = 0$

$$u_2'(x) = \frac{W_2}{W} = \frac{y_1(x) f(x)}{W}$$

$$u_2(x) = \int u_2'(x)\, dx$$

$$\text{anti-derivative} = \int \frac{y_1(t) f(t)}{W(t)}\, dt + c_2$$

$$= \int_{x_0}^{x} \frac{y_1(t) f(t)}{W(t)}\, dt$$

$$u_2(x_0) = 0$$

➡ $u_2(x_0)\, y_2(x_0) = 0$

➡ $u_2(x_0)\, y_2'(x_0) = 0$

Zero Initial Conditions

$$u_1(x_0) = 0 \Rightarrow u_1(x_0)\,y_1(x_0) = 0 \qquad \Rightarrow u_1(x_0)\,y_1'(x_0) = 0$$

$$u_2(x_0) = 0 \Rightarrow u_2(x_0)\,y_2(x_0) = 0 \qquad \Rightarrow u_2(x_0)\,y_2'(x_0) = 0$$

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x) \Rightarrow y_p(x_0) = u_1(x_0)\,y_1(x_0) + u_2(x_0)\,y_2(x_0) = 0$$

$$y_p'(x) = u_1'(x)\,y_1(x) + u_1(x)\,y_1'(x) + u_2'(x)\,y_2(x) + u_2(x)\,y_2'(x)$$

$$\Rightarrow y_p'(x_0) = u_1'(x_0)\,y_1(x_0) + u_1'(x_0)\,y_1(x_0) = -\frac{y_1(x_0)\,y_2(x_0)\,f(x_0)}{W} + \frac{y_1(x_0)\,y_2(x_0)\,f(x_0)}{W} = 0$$

$$y_p(x) = u_1(x)\,y_1 + u_2(x)\,y_2 \Rightarrow y_p(x_0) = 0 \qquad y_p'(x_0) = 0$$

$$y_p(x) = y_1(x)\int_{x_0}^{x} -\frac{y_2(t)\,f(t)}{W(t)}\,dt + y_2(x)\int_{x_0}^{x} \frac{y_1(t)\,f(t)}{W(t)}\,dt \Rightarrow y_p(x_0) = 0 \qquad y_p'(x_0) = 0$$

Zero State Solution

$$y_p(x) = y_1(x)\int_{x_0}^{x} -\frac{y_2(t)f(t)}{W(t)}\,dt + y_2(x)\int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}\,dt \Rightarrow \quad y_p(x_0) = 0 \qquad y_p'(x_0) = 0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y'(x_0) = 0$$

$$y(x_0) = 0$$

$$y_p(x) = y_1(x)\int_{x_0}^{x} -\frac{y_2(t)f(t)}{W(t)}\,dt + y_2(x)\int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}\,dt$$

Green's Function and IVP's (1)

$$y'' + P(x)\,y' + Q(x)\,y = f(x)$$

$$[x_0,\ x] \subset I$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(x)$$

$$u_1'(x) = -\frac{y_2(x) f(x)}{W(x)} \qquad u_1(x) = \int_{x_0}^{x} -\frac{y_2(t) f(t)}{W(t)}\,dt$$

$$u_2'(x) = \frac{y_1(x) f(x)}{W(x)} \qquad u_2(x) = \int_{x_0}^{x} \frac{y_1(t) f(t)}{W(t)}\,dt$$

$$y_p = u_1(x)\,y_1 + u_2(x)\,y_2$$

$$= \left[\int_{x_0}^{x} -\frac{y_2(t) f(t)}{W(t)}\,dt\right] y_1(x) + \left[\int_{x_0}^{x} \frac{y_1(t) f(t)}{W(t)}\,dt\right] y_2(x)$$

$$= \left[\int_{x_0}^{x} -\frac{y_1(x)\,y_2(t)}{W(t)} f(t)\,dt\right] + \left[\int_{x_0}^{x} \frac{y_1(t)\,y_2(x)}{W(t)} f(t)\,dt\right]$$

$$= \int_{x_0}^{x} \left[\frac{y_1(t)\,y_2(x) - y_1(x)\,y_2(t)}{W(t)}\right] f(t)\,dt$$

$$= \int_{x_0}^{x} G(x,\ t)\, f(t)\,dt$$

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Green's Function and IVP's (2)

$$y'' + P(x)y' + Q(x)y = 0$$
$$y'(x_0) = y_1$$
$$y(x_0) = y_0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y'(x_0) = 0$$
$$y(x_0) = 0$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2 = \int_{x_0}^{x} \left[\frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{W(t)} \right] f(t)\, dt = \int_{x_0}^{x} G(x, t) f(t)\, dt$$

at the end, this x will replace the literal t

$$= \int_{x_0}^{x} G(x, t) f(t)\, dt$$

this x and t appear in the indefinite integral

Green's Function

$$G(x,\ t) = \left[\frac{y_1(t)\, y_2(x) - y_1(x)\, y_2(t)}{W(t)} \right] \qquad W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

$$y'' + P(x)\,y' + Q(x)\,y = 0 \quad \rightarrow \quad y_{1},\ \ y_2$$

$$y'' + P(x)\,y' + Q(x)\,y = f(x)$$

$$y'' + P(x)\,y' + Q(x)\,y = g(x)$$

$$y'' + P(x)\,y' + Q(x)\,y = h(x)$$

the same Green's function

$$G(x,\ t) = \left[\frac{y_1(t)\, y_2(x) - y_1(x)\, y_2(t)}{W(t)} \right]$$

Three Initial Value Problem

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

$$y'' + P(x)y' + Q(x)y = 0$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

Homogeneous DEQ

Nonhomogeneous Initial Conditions

Nonzero Initial Conditions

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

Nonhomogeneous DEQ

Zero Initial Conditions

Initially at rest

Rest Solution

General Solutions of the Initial Value Problem

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

$$y = y_h + y_p$$

$$y(x_0) = y_h(x_0) + y_p(x_0) = y_0 + 0 = y_0$$
$$y'(x_0) = y_h'(x_0) + y_p'(x_0) = y_1 + 0 = y_1$$

$$y'' + P(x)y' + Q(x)y = 0$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

$$y_h$$

Nonhomogeneous Initial Conditions

Nonzero Initial Conditions

Response due to the initial conditions

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

$$y_p = \int_{x_0}^{x} G(x, t) f(t)\, dt$$

Response due to the forcing function f

Zero Initial Conditions

Initially at rest

Rest Solution

Rest Solution

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

Nonhomogeneous DEQ

Zero Initial Conditions

Initially at rest

Rest Solution

$$y_p = u_1(x)y_1 + u_2(x)y_2 = \int_{x_0}^{x} \left[\frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}\right] f(t)\,dt = \int_{x_0}^{x} G(x, t) f(t)\,dt$$

$$y_p(x) = \int_{x_0}^{x} G(x, t) f(t)\,dt$$

$$y_p'(x) = G(x, x) f(x) + \int_{x_0}^{x} \frac{\partial}{\partial x}[G(x, t) f(t)]\,dt = \int_{x_0}^{x} \left[\frac{y_1(t)y_2'(x) - y_1'(x)y_2(t)}{W(t)}\right] f(t)\,dt$$

$$y_p(x_0) = \int_{x_0}^{x_0} G(x, t) f(t)\,dt = 0$$

$$y_p'(x_0) = \int_{x_0}^{x_0} \left[\frac{y_1(t)y_2'(x_0) - y_1'(x_0)y_2(t)}{W(t)}\right] f(t)\,dt = 0$$

Green's Function

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y'' + P(x)y' + Q(x)y = g(x)$$

$$y'' + P(x)y' + Q(x)y = h(x)$$

$y_1, \; y_2$

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

the same Green's function

$$G(x, t) = \left[\frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{W(t)} \right]$$

$$y_p = u_1(x) y_1 + u_2(x) y_2 = \int_{x_0}^{x} G(x, t) f(t)\, dt = \int_{x_0}^{x} \left[\frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{W(t)} \right] f(t)\, dt$$

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General Solutions of the Initial Value Problem

$$y'' + 5y' + 6y = f(x)$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

$$m^2 + 5m + 6 = (m+2)(m+3) = 0$$

$$m = -2, \ -3$$

$$y_1 = e^{-2t} \qquad y_2 = e^{-3t}$$

$$W(t) = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$

$$G(x, t) = \left[\frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{W(t)} \right]$$

$$= \left[\frac{e^{-2t} e^{-3x} - e^{-2x} e^{-3t}}{-e^{-5t}} \right]$$

$$= \left[-e^{3t} e^{-3x} + e^{-2x} e^{+2t} \right]$$

$$= \left[e^{-2(x-t)} - e^{-3(x-t)} \right]$$

$$= h(x-t)$$

$$y_p = \int_{x_0}^{x} h(x - t) f(t) dt$$

Impulse Response by the Green's function

$$y'' + 3y' + 2y = x'$$

$$y'' + 3y' + 2y = x$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

$$m^2+3m+2 = (m+1)(m+2) = 0$$

$$m = -1, \ -2$$

$$y_1 = e^{-t} \qquad y_2 = e^{-2t}$$

$$W(t) = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -e^{-3t}$$

$$G(x, t) = \left[\frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}\right]$$

$$= \left[\frac{e^{-t}e^{-2x} - e^{-1x}e^{-2t}}{-e^{-3t}}\right]$$

$$= \left[-e^{2t}e^{-2x} + e^{-x}e^{+t}\right]$$

$$= \left[e^{-(x-t)} - e^{-2(x-t)}\right]$$

$$= y_n(x-t)$$

$$y_p = \int_{x_0}^{x} h(x-t)f(t)\,dt$$

$$y_n(t) = \left[e^{-t} - e^{-2t}\right]$$

$$h(t) = [D\,y_n(t)]u(t)$$

$$= -e^{-t} + 2e^{-2t}$$

References

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